Infinite products of 2×2 matrices and the Gibbs properties of Bernoulli convolutions

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Abstract.– We consider the infinite sequences $(A_n)_{n\in\mathbb{N}}$ of 2×2 matrices with nonnegative entries, where the A_n are taken in a finite set of matrices. Given a vector $V=\begin{pmatrix}v_1\\v_2\end{pmatrix}$ with $v_1,v_2>0$, we give a necessary and sufficient condition for $\dfrac{A_1\dots A_nV}{||A_1\dots A_nV||}$ to converge uniformly. In application we prove that the Bernoulli convolutions related to the numeration in Pisot quadratic bases are weak Gibbs.

Key-words: Infinite products of matrices, weak Gibbs measures, Bernoulli convolutions, Pisot numbers, β-numeration.

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Introduction

Let $\mathcal{M}=\{M_0,\dots,M_{\mathbf{s}-1}\}$ be a finite subset of the set – stable by matrix multiplication – of nonnegative and column-allowable $d\times d$ matrices (i.e., the matrices with nonnegative entries and without null column). We associate to any sequence $(\omega_n)_{n\in\mathbb{N}}$ with terms in $\mathcal{S}:=\{0,1,\dots,\mathbf{s}-1\}$, the sequence of product matrices

$$P_n(\omega)=M_{\omega_1}M_{\omega_2}\dots M_{\omega_n}.$$

Experimentally, in most cases each normalized column of $P_n(\omega)$ converges when $n\to\infty$ to a limit-vector, which depends on $\omega\in\mathcal{S}^{\mathbb{N}}$ and may depend on the index of the column. Nevertheless the normalized rows of $P_n(\omega)$ in general do not converge: suppose for instance that all the matrices in $\mathcal{M}$ are positive but do not have the same positive normalized left-eigenvector, let L_k such that $L_kM_k=\rho_kL_k$. For any positive matrix M, the normalized rows of $M{M_0}^n$ converge to L_0 and the ones of $M{M_1}^n$ to L_1. Consequently we can choose the sequence $(n_k)_{k\in\mathbb{N}}$ sufficiently increasing such that the normalized rows of $M_0^{n_1}M_1^{n_2}\dots M_0^{n_{2k-1}}$ converge to L_0 while the ones of $M_0^{n_1}M_1^{n_2}\dots M_0^{n_{2k-1}}M_1^{n_{2k}}$ converge to L_1. This proves – if $L_0\neq L_1$ – that the normalized rows in $P_n(\omega)$ do not converge when $\omega=0^{n_1}1^{n_2}0^{n_3}1^{n_4}\dots$.

Now in case $\mathcal{M}$ is a set of positive matrices it is clear that, if both normalized columns and normalized rows in $P_n(\omega)$ converge then – after replacing each matrix M_k by $\frac{1}{\rho_k} M_k$ – the matrix $P_n(\omega)$ itself converges: the previous counterexample proves that the matrices $P_n(\omega)$ have a common left-eigenvector for any n, and a straightforward computation (using the limits of the normalized columns in $P_n(\omega)$) proves the existence of $\lim\limits_{n\to\infty} P_n(\omega)$.

The existence of a common left-eigenvector is settled in a more general context by L. Elsner and S. Friedland ([5, Theorem 1]), in case $\mathcal{M}$ is a finite set of matrices with entries in $\mathbb{C}$. This theorem means (after transposition of the matrices) that if $P_n(\omega)$ converges to a non-null limit, then there exists $N \in \mathbb{N}$ such that the matrices M_{ω_n} for $n \geq N$ have a common left-eigenvector for the eigenvalue 1. Now, L. Elsner & S. Friedland (in [5, Main Theorem]) and I. Daubechies & J. C. Lagarias (in [2, Theorem 5.1] (resp. [1, Theorem 4.2])) give necessary and sufficient conditions for $P_n(\omega)$ to converge for any $\omega \in \mathcal{S}^{\mathbb{N}}$ (resp., to converge to a continuous map).

By these theorems we see that the problem of the convergence of the normalized columns in $P_n(\omega)$ is very different from the problem of the convergence of $P_n(\omega)$ itself. Let for instance $M_0 = \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$ and $M_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$; then the normalized columns in

$$P_n(\omega) = \begin{cases} \dfrac{1}{10} \cdot \begin{pmatrix} 4+6\cdot 6^{-n} & 6-6\cdot 6^{-n} \\ 4-4\cdot 6^{-n} & 6+4\cdot 6^{-n} \end{pmatrix} & \text{if } \omega_1 \dots \omega_n = 0\dots 0 \\ \dfrac{1}{10} \cdot \begin{pmatrix} 4+6^{-h} & 6-6^{-h} \\ 4+6^{-h} & 6-6^{-h} \end{pmatrix} & \text{if } \omega_1 \dots \omega_n = \omega_1 \dots \omega_{n-h-1} 10 \dots 0 \end{cases}$$

converge to $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ for any $\omega \in \{0,1\}^{\mathbb{N}}$, but $P_n(\omega)$ diverges (altough it is bounded) if ω is not eventually constant.

In Section 1 we study the uniform convergence – in direction – of $P_n(\omega)V$ in case the M_k are 2×2 nonnegative column-allowable matrices and $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ a positive vector (Theorem 1.1). Notice that the convergence in direction of the columns of $P_n(\omega)$, to a same vector, implies the ones of $P_n(\omega)V$, but the converse is not true: see for instance the case $\mathcal{M} = \left\{ \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \right\}$.

The second section is devoted to the *Bernoulli convolutions* [4], which have been studied since the early 1930's (see [8] for the other references). We give a matricial relation for such measures.

In the third section we apply more precisely Theorem 1.1 to prove that certain Bernoulli convolutions are weak Gibbs in the following sense (see [10]): given a system off affine contractions $\mathbb{S}_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that the intervals $\mathbb{S}_\varepsilon([0,1])$ make a partition of $[0,1]$ for $\varepsilon \in \mathcal{S} = \{0, 1, \ldots, \mathtt{s}-1\}$, a measure η supported by $[0,1]$ is weak Gibbs w.r.t. $\{\mathbb{S}_\varepsilon\}_{\varepsilon=0}^{\mathtt{s}-1}$ if there exists a map $\Phi : \mathcal{S}^{\mathbb{N}} \to \mathbb{R}$, continuous for the product topology, such that

$$\lim_{n\to\infty} \left(\frac{\eta[\![\xi_1 \ldots \xi_n]\!]}{\exp\left(\sum_{k=0}^{n-1} \Phi(\sigma^k \xi)\right)} \right)^{1/n} = 1 \quad \text{uniformly on } \xi \in \mathcal{S}^{\mathbb{N}}, \tag{1}$$

where $[\![\xi_1 \ldots \xi_n]\!] := \mathbb{S}_{\xi_1} \circ \ldots \circ \mathbb{S}_{\xi_n}([0,1])$ and σ is the shift on $\mathcal{S}^{\mathbb{N}}$. Let us give a sufficient condition for η to be weak Gibbs. For each $\xi \in \mathcal{S}^{\mathbb{N}}$ we put $\phi_1(\xi) = \log \eta[\![\xi_1]\!]$ and for $n \geq 2$,

$$\phi_n(\xi) = \log \left(\frac{\eta[\![\xi_1 \cdots \xi_n]\!]}{\eta[\![\xi_2 \cdots \xi_n]\!]} \right). \tag{2}$$

The continuous map $\phi_n : \mathcal{S}^{\mathbb{N}} \to \mathbb{R}$ ($n \geq 1$) is the *n-step potential* of η. Assume the existence of the uniform limit $\Phi = \lim_{n\to\infty} \phi_n$; it is then straightforward that for $n \geq 1$,

$$\frac{1}{K_n} \leq \frac{\eta[\![\xi_1 \cdots \xi_n]\!]}{\exp\left(\sum_{k=0}^{n-1} \Phi(\sigma^k \xi)\right)} \leq K_n \quad \text{with} \quad K_n = \exp\left(\sum_{k=1}^{n} \|\Phi - \phi_n\|_\infty \right). \tag{3}$$

By a well known lemma on the Cesàro sums, $K_1, K_2, \ldots$ form a subexponential sequence of positive real numbers, that is $\lim_{n\to\infty} (K_n)^{1/n} = 1$ and thus, (3) means η is weak Gibbs w.r.t. $\{\mathbb{S}_\varepsilon\}_{\varepsilon=0}^{\mathtt{s}-1}$.

Now the weak Gibbs property can be proved for certain Bernoulli convolutions by computing the n-step potential by means of products of matrices (see [6] for the Bernoulli convolution associated with the golden ratio $\beta = \dfrac{1+\sqrt{5}}{2}$ – called the *Erdös measure* – and the application to the multifractal analysis). In Theorem 3.1 we generalize this result in case $\beta > 1$ is a quadratic number with conjugate $\beta' \in]-1, 0[$.

1 Infinite product of 2×2 matrices

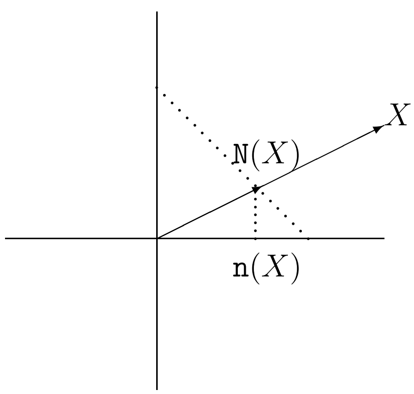
From now the vectors $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and the matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we consider are supposed to be nonnegative and column-allowable that is, x_1, x_2, a, b, c, d are nonnegative

and $x_1+x_2, a+c, b+d$ are positive. In particular we suppose that the matrices in $\mathcal{M}=\{M_0,\ldots,M_{\mathtt{s}-1}\}$ satisfy these conditions. We associate to X the normalized vector:

$$\mathtt{N}(X) := \begin{pmatrix} \frac{x_1}{x_1+x_2} \\ \frac{x_2}{x_1+x_2} \end{pmatrix} = \begin{pmatrix} \mathtt{n}(X) \\ 1-\mathtt{n}(X) \end{pmatrix} \quad \text{where} \quad \mathtt{n}(X) := \frac{x_1}{x_1+x_2}$$

and define the distance between the column of A (or the rows of ${}^tA$):

$$d_{\text{columns}}(A) := \left| \mathtt{n}\left(\begin{pmatrix} a \\ c \end{pmatrix}\right) - \mathtt{n}\left(\begin{pmatrix} b \\ d \end{pmatrix}\right) \right| = \frac{|\det A|}{(a+c)(b+d)} =: d_{\text{rows}}({}^tA).$$



Theorem 1.1 *Given* $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *with* $v_1, v_2 > 0$, *the sequence of vectors* $\mathtt{N}(P_n(\omega)V)$ *converges uniformly for* $\omega \in \mathcal{S}^{\mathbb{N}}$ *only in the five following cases:*

Case 1: $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{M} \Rightarrow a \geq d$; $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathcal{M} \Rightarrow a \leq d$;

$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M} \Rightarrow bc' \geq b'c$; *no matrix in* $\mathcal{M}$ *has the form* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ *or* $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$.

Case 2: $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{M} \Rightarrow a < d$; $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathcal{M} \Rightarrow a > d$;

$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M} \Rightarrow bc' < b'c$.

Case 3: $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{M} \Rightarrow a \geq d$; $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathcal{M} \Rightarrow a > d$;

no matrix in $\mathcal{M}$ *has the form* $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$.

Case 4: $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{M} \Rightarrow a < d$; $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathcal{M} \Rightarrow a \leq d$;

no matrix in $\mathcal{M}$ has the form $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$*;*

Case 5: V is an eigenvector of all the matrices in $\mathcal{M}$.

COROLLARY 1.2 *If $\mathtt{N}(P_n(\cdot)V)$ converges uniformly on $\mathcal{S}^{\mathbb{N}}$, the limit do not depend on the positive vector V, except in the fifth case of Theorem 1.1.*

Proof. Suppose that $\mathcal{M}$ satisfies the conditions of the case 1,2,3 or 4 in Theorem 1.1 and let V, W be two positive vectors. Then the following set $\mathcal{M}'$ also do:

$\mathcal{M}' := \mathcal{M} \cup \{M_s\}$, where M_s is the matrix whose both columns are W.

Denoting by $\omega' = \omega_1 \dots \omega_n \overline{s}$ the sequence defined by $\omega'_i = \begin{cases} \omega_i & \text{if } i \le n \\ s & \text{if } i > n \end{cases}$ one has for any $\omega \in \mathcal{S}^{\mathbb{N}}$

$$\mathtt{N}(P_n(\omega)V) - \mathtt{N}(P_n(\omega)W) = \mathtt{N}(P_n(\omega')V) - \mathtt{N}(P_{n+1}(\omega')V)$$

and this tends to 0, according to the uniform Cauchy property of the sequence $\mathtt{N}(P_n(\cdot)V)$. ■

Nevertheless, this limit may depend of V if one assume only that V is nonnegative. For instance, if $\mathcal{M} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ then $\lim_{n\to\infty} \mathtt{N}\left(P_n(\omega) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ differs from $\lim_{n\to\infty} \mathtt{N}\left(P_n(\omega) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$ iff $\omega = \overline{0}$ (implying the second limit is not uniform on $\mathcal{S}^{\mathbb{N}}$).

1.1 Geometric considerations

We follow the ideas of E. Seneta about products of nonnegative matrices in Section 3 of [9], or stochastic matrices in Section 4. In what follows we denote the matrices by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ or $A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ for $n \in \mathbb{N}$, and we suppose they are nonnegative and column-allowable. We define the coefficient

$$\tau(A) := \sup_{d_{\text{columns}}(A') \neq 0} \frac{d_{\text{columns}}(A'A)}{d_{\text{columns}}(A')}.$$

The straightforward formula

$$d_{\text{columns}} \left(\prod_{k=1}^{n} (A_k) \right) \le d_{\text{columns}}(A_1) \prod_{k=2}^{n} \tau(A_k) \tag{4}$$

is of use to prove Theorem 1.1 because, according to the following proposition one has $\tau(A) < 1$ if A is positive.

Proposition 1.3

$$\tau(A) = \begin{cases} \dfrac{\left|\sqrt{ad}-\sqrt{bc}\right|}{\sqrt{ad}+\sqrt{bc}} & \textit{if } A \textit{ do not have any null row} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. $\dfrac{d_{\text{columns}}(A'A)}{d_{\text{columns}}(A')} = \dfrac{|\det A|}{(a+c/x)(bx+d)}\left(\text{where } x = \dfrac{a'+c'}{b'+d'}\right)$ is maximal for $x = \sqrt{\dfrac{cd}{ab}}$. ■

Remark 1.4 *One can consider – instead of* d_{columns} *– the angle between the columns of* A*:*

$$\alpha(A) := \left|\arctan\frac{a}{c} - \arctan\frac{b}{d}\right|,$$

or the Hilbert distance between the columns of a positive matrice A*:*

$$d_{\text{Hilbert}}(A) := \left|\log\frac{a}{c} - \log\frac{b}{d}\right|.$$

This last can be interpreted either as the distance between the columns or the rows of A*, because* $d_{\text{Hilbert}}(A) = d_{\text{Hilbert}}({}^tA)$*. The Birkhoff coefficient* $\tau_{\text{Birkhoff}}(A) := \sup\limits_{d_{\text{Hilbert}}(A')\neq 0} \dfrac{d_{\text{Hilbert}}(A'A)}{d_{\text{Hilbert}}(A')}$ *has – from [9, Theorem (3.12)] – the same value as* $\tau(A)$ *in Proposition 1.3, and probably as a large class of coefficients defined in this way.*

In the following proposition we list the properties of d_{columns} that are required for proving Theorem 1.1.

Proposition 1.5 *(i)* $\sup\limits_{d_{\text{columns}}(A')\neq 0} \dfrac{d_{\text{columns}}(AA')}{d_{\text{columns}}(A')} = \dfrac{|\det A|}{\min\left((a+c)^2,(b+d)^2\right)} =: \tau_1(A).$

(ii) If A *is positive then* $\sup\limits_{d_{\text{columns}}(A')\neq 0} \dfrac{d_{\text{rows}}(AA')}{d_{\text{columns}}(A')} \leq \dfrac{|\det A|}{\min(a,b)\cdot\min(c,d)} =: \tau_2(A).$

(iii) If $\lim\limits_{n\to\infty} d_{\text{columns}}(A_n) = 0$ *then* $\lim\limits_{n\to\infty} d_{\text{columns}}(AA_nA') = 0$ *and, assuming that* A *is positive,* $\lim\limits_{n\to\infty} d_{\text{rows}}(AA_nA') = 0$.

(iv) Suppose the matrices A_n *are upper-triangular. If* $\inf\limits_{k\in\mathbb{N}} \dfrac{a_k}{d_k} \geq 1$ *and* $\sum\limits_{k\in\mathbb{N}} \dfrac{b_k}{d_k} = \infty$ *then*

$$\lim_{n\to\infty} d_{\text{columns}}(A_1\dots A_n) = \lim_{n\to\infty} d_{\text{columns}}(A_n\dots A_1) = 0.$$

(v) Suppose the matrices A_n *are lower-triangular. If* $\inf\limits_{k\in\mathbb{N}} \dfrac{d_k}{a_k} \geq 1$ *and* $\sum\limits_{k\in\mathbb{N}} \dfrac{c_k}{a_k} = \infty$ *then*

$$\lim_{n\to\infty} d_{\text{columns}}(A_1\dots A_n) = \lim_{n\to\infty} d_{\text{columns}}(A_n\dots A_1) = 0.$$

Proof. (i) and (ii) are obtained from the formula

$$d_{\text{columns}}(AA') = \frac{\det A \cdot \det A'}{((a+c)a' + (b+d)c') \cdot ((a+c)b' + (b+d)d')},$$

and the relation $d_{\text{rows}}(AA') = d_{\text{columns}}({}^tA'\ {}^tA)$.

(iii) is due to the fact that the inequalities of items (i), (ii) and (4) imply $d_{\text{columns}}(AA_nA') \leq \tau_1(A) d_{\text{columns}}(A_n)\tau(A')$ and – if A is positive – $d_{\text{rows}}(AA_nA') \leq \tau_2(A) d_{\text{columns}}(A_n)\tau(A')$.

(iv) follows from the formula

$$A_1 \dots A_n = \begin{pmatrix} a_1 \dots a_n & s_n \\ 0 & d_1 \dots d_n \end{pmatrix},$$

where $s_n = \sum_{k=1}^{n} a_1 \dots a_{k-1} b_k d_{k+1} \dots d_n \geq d_1 \dots d_n \sum_{k=1}^{n} \frac{b_k}{d_k}$.

(v) can be deduced from (iv) by using the relation $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ 0 & a \end{pmatrix}$. ■

We need also the following:

Proposition 1.6 *Let V_A be a nonnegative eigenvector associated to the maximal eigenvalue of A, and C a cone of nonnegative vectors containing V_A. If $\det A \geq 0$ then C is stable by left-multiplication by A.*

Proof. The discriminant of the characteristic polynomial of A is $(a-d)^2 + 4bc$. In case this discriminant is null the proof is obtained by direct computation, because $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ or $\begin{pmatrix} a & 0 \\ c & a \end{pmatrix}$. Otherwise A has two eigenvalues $\lambda > \lambda'$ and, given a nonnegative vector X, there exists a real α and an eigenvector W_A (associated to λ') such that

$$X = \alpha V_A + W_A \quad \text{and} \quad AX = \lambda\alpha V_A + \lambda' W_A = \lambda' X + (\lambda - \lambda')\alpha V_A.$$

Notice that $\alpha \geq 0$ (because the nonnegative vector $A^n X = \lambda^n \alpha V_A + \lambda'^n W_A$ converges in direction to αV_A) and $\lambda' \geq 0$ (from the hypothesis $\det A \geq 0$). Hence AX is a nonnegative linear combination of X and V_A; if X belongs to C then AX also do. ■

1.2 How pointwise convergence implies uniform convergence

Let m and M be the bounds of $\mathtt{n}(P_n(\omega)V)$ for $n \in \mathbb{N}$ and $\omega \in \mathcal{S}^{\mathbb{N}}$, and let $M_V := \begin{pmatrix} m & M \\ 1-m & 1-M \end{pmatrix}$. Each real $x \in [m, M]$ can be written $x = mx_1 + Mx_2$ with $x_1, x_2 \geq 0$

and $x_1 + x_2 = 1$; in particular the real $x = \mathtt{n}(P_n(\omega)V)$ can be written in this form, hence

$$\forall \omega \in \mathcal{S}^{\mathbb{N}},\ \exists t_1, t_2 \geq 0,\ P_n(\omega)V = M_V \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}. \tag{5}$$

PROPOSITION 1.7 *If $d_{\text{columns}}(P_n(\cdot)M_V)$ converges pointwise to 0 when $n \to \infty$, then $\mathtt{N}(P_n(\cdot)V)$ converges uniformly on $\mathcal{S}^{\mathbb{N}}$.*

Proof. Suppose the pointwise convergence holds. Given $\omega \in \mathcal{S}^{\mathbb{N}}$ and $\varepsilon > 0$, there exists the integer $n = n(\omega, \varepsilon)$ such that $d_{\text{columns}}(P_n(\omega)M_V) \leq \varepsilon$. The family of cylinders $C(\omega, \varepsilon) := [\![\omega_1 \ldots \omega_{n(\omega,\varepsilon)}]\!]$, for ω running over $\mathcal{S}^{\mathbb{N}}$, is a covering of the compact $\mathcal{S}^{\mathbb{N}}$; hence there exists a finite subset $X \subset \mathcal{S}^{\mathbb{N}}$ such that $\mathcal{S}^{\mathbb{N}} = \bigcup_{\omega \in X} C(\omega, \varepsilon)$. Let $p > q \geq n_\varepsilon := \max\{n(\omega, \varepsilon)\ ;\ \omega \in\ X\}$. For any $\xi \in \mathcal{S}^{\mathbb{N}}$, there exists $\zeta \in X$ such that $\xi \in C(\zeta, \varepsilon)$ that is, $\xi_k = \zeta_k$ for any $k \leq n = n(\zeta, \varepsilon)$. From (5) there exists two nonnegative vectors V_p and V_q such that $P_p(\xi)V = P_n(\zeta)M_V V_p$ and $P_q(\xi)V = P_n(\zeta)M_V V_q$. Denoting by $M(p,q)$ the column-allowable matrix whose columns are V_p and V_q we have – in view of (4)

$$\begin{aligned} |\mathtt{n}\,(P_p(\xi)V) - \mathtt{n}\,(P_q(\xi)V)| &= d_{\text{columns}}\,(P_n(\zeta)M_V M(p,q)) \\ &\leq d_{\text{columns}}\,(P_n(\zeta)M_V) \\ &\leq \varepsilon, \end{aligned}$$

implying the uniform Cauchy property for $\mathtt{N}(P_n(\cdot)V)$. ■

1.3 Proof of the uniform convergence of $\mathtt{N}(P_n(\cdot)V)$

According to Proposition 1.7 it is sufficient to prove that $\lim\limits_{n\to\infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$ for each $\omega \in \mathcal{S}^{\mathbb{N}}$. This convergence is obvious in the following cases:

• If there exists N such that M_{ω_N} has rank 1, then $P_n(\omega)M_V$ has rank 1 for $n \geq N$ and

$$\forall n \geq N,\ d_{\text{columns}}(P_n(\omega)M_V) = 0.$$

• If there exists infinitely many integers n such that M_{ω_n} is a positive matrix, one has $\tau(M_{\omega_n}) \leq \rho := \max\limits_{M \in \mathcal{M},\ M>0} \tau(M) < 1$, and the formula (4) implies

$$\lim_{n\to\infty} d_{\text{columns}}(P_n(\omega)M_V) = 0.$$

• Similarly, this limit is null also in case there exists infinitely many integers n such that $M_{\omega_n} M_{\omega_{n+1}}$ is a positive matrix.

So we can make from now the following hypotheses on the sequence ω under consideration:

(H): $\det M_{\omega_n} \neq 0$ for any $n \in \mathbb{N}$, and there exists N such that the matrix $M_{\omega_n} M_{\omega_{n+1}}$ has at least one null entrie for any $n > N$.

Proof in the case 1: Since the couples of matrices $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}$ satisfy $\dfrac{b}{c} \geq \dfrac{b'}{c'}$, there exists a real α such that

$$\forall \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}, \quad \frac{b}{c} \geq \alpha \geq \frac{b'}{c'}.$$

Let $\Delta = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$. We denote by $\mathcal{P}$ the set of 2×2 matrices with nonnegative determinant and by $\tilde{\mathcal{M}}$ the subset of $\mathcal{P}$ defined as follows:

$$\tilde{\mathcal{M}} := \{\Delta^{-1}M,\ M\Delta\ ;\ M \in \mathcal{M} \setminus \mathcal{P}\} \cup \{M,\ \Delta^{-1}M\Delta\ ;\ M \in \mathcal{M} \cap \mathcal{P}\}.$$

This set of matrices also satisfies the conditions mentionned in the case 1: for instance if $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in \mathcal{M}$, the matrix $\Delta^{-1}\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} c & 0 \\ a/\alpha & b/\alpha \end{pmatrix}$ satisfies $c \leq b/\alpha$, and so one. For any sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ of elements of $\{0, 1\}$ such that $\varepsilon_0 = 0$ we can write

$$\begin{array}{rcl} P_n(\omega) & = & M_{\omega_1} M_{\omega_2} \ldots M_{\omega_n} \\ & = & (\Delta^{-\varepsilon_0} M_{\omega_1} \Delta^{\varepsilon_1}) \cdot (\Delta^{-\varepsilon_1} M_{\omega_2} \Delta^{\varepsilon_2}) \cdot \ldots \cdot (\Delta^{-\varepsilon_{n-1}} M_{\omega_n} \Delta^{\varepsilon_n}) \cdot \Delta^{-\varepsilon_n} \\ & = & A_1 A_2 \ldots A_n \Delta^{-\varepsilon_n} \end{array} \tag{6}$$

where $A_n := \Delta^{-\varepsilon_{n-1}} M_{\omega_n} \Delta^{\varepsilon_n}$ for any $n \in \mathbb{N}$. By the following choice of the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, the matrices A_n belong to $\tilde{\mathcal{M}}$:

$$\varepsilon_n = \begin{cases} \varepsilon_{n-1} & \text{if } \det M_{\omega_n} > 0 \\ 1 - \varepsilon_{n-1} & \text{otherwise.} \end{cases}$$

The hypotheses (H) imply that either all the matrices A_n for $n > N$ are upper-triangular, or all of them are lower-triangular (otherwise $M_{\omega_n} M_{\omega_{n+1}} = \Delta^{\varepsilon_{n-1}} A_n A_{n+1} \Delta^{-\varepsilon_{n+1}}$ is positive for some $n > N$). By Proposition 1.5 (iv) and (v),

$$\lim_{n\to\infty} d_{\text{columns}}(A_{N+1} \ldots A_n) = 0.$$

From (6) and Proposition 1.5 (iii), $\lim\limits_{n\to\infty} d_{\text{columns}}(P_n(\omega) M_V) = 0$.

Proof in the case 2: We use the matrix Δ and the set of matrices $\tilde{\mathcal{M}}$ defined in the previous case; here the real α is supposed such that $\dfrac{b}{c} \leq \alpha \leq \dfrac{b'}{c'}$ for any $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix} \in \mathcal{M}$,

and consequently $\tilde{\mathcal{M}}$ satisfies the hypotheses of the case 2. This imply that each matrix in $\tilde{\mathcal{M}}$ has a positive eigenvector. Let C be the (minimal) cone containing V, $\Delta^{-1}V$ and the positive eigenvectors of the matrices in $\tilde{\mathcal{M}}$. From (6) and Proposition 1.6, $P_n(\omega)V$ belongs to this cone for any $\omega \in \mathcal{S}^{\mathbb{N}}$ hence M_V is positive.

Using again the relation (6) we have

$$d_{\text{columns}}\left(P_n(\omega)M_V\right) = d_{\text{rows}}\left({}^tM_V\ {}^t\Delta^{-\varepsilon_n}\ {}^tA_n \dots {}^tA_1\right). \tag{7}$$

Each matrix ${}^tA_n$ for $n > N$ satisfy $a > d$ if ${}^tA_n$ is upper-triangular, and $a < d$ if it is lower-triangular. By Proposition 1.5 (iv) and (v), $\lim_{n\to\infty} d_{\text{columns}}({}^tA_n \dots {}^tA_{N+1}) = 0$. This implies $\lim\limits_{n\to\infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$ by applying Proposition 1.5 (iii) to the r.h.s. in (7).

Proof in the case 3: Let C' be the (minimal) cone containing V, the nonnegative eigenvectors (associated to the maximal eigenvalues) of the matrices in $\mathcal{M}\cap\mathcal{P}$, and the column-vectors of the matrices in $\mathcal{M}\setminus\mathcal{P}$. All the vectors delimiting C' are distinct from $\begin{pmatrix}0\\1\end{pmatrix}$, and Proposition 1.6 implies that $P_n(\omega)V \in C'$ for any $\omega \in \mathcal{S}^{\mathbb{N}}$. Hence m and M that is, the bounds of $\mathtt{n}(P_n(\omega)V))$, are positive.

Suppose first that M_{ω_n} is lower-triangular for any $n \in \mathbb{N}$ and let $\begin{pmatrix}\alpha_n & 0\\ \gamma_n & \delta_n\end{pmatrix} = P_n(\omega)$. The hypotheses of the case 3 imply $\lim\limits_{n\to\infty} \dfrac{\delta_n}{\alpha_n} = 0$. A simple computation gives $d_{\text{columns}}\left(P_n(\omega)M_V\right) \leq \dfrac{\delta_n}{\alpha_n}\cdot\dfrac{M-m}{Mm}$ hence $\lim\limits_{n\to\infty} d_{\text{columns}}\left(P_n(\omega)M_V\right) = 0$. This conclusion remains valid if M_{ω_n} is eventually lower-triangular.

Suppose now M_{ω_n} is not lower-triangular for infinitely many n. The hypotheses mentionned in the case 3 and (H) imply that M_{ω_n} is upper-triangular for any $n > N$ (because for each n such that M_{ω_n} is lower-triangular or has the form $\begin{pmatrix}a & b\\ c & 0\end{pmatrix}$, (H) implies that $M_{\omega_{n+1}}$ is lower-triangular). Proposition 1.5 (iii) and(iv) implies that $\lim\limits_{n\to\infty} d_{\text{columns}}(P_n(\omega)M_V) = 0$.

Proof in the case 4: Let $\mathcal{M}'$ be the set of matrices $M'_k = \Delta^{-1}M_k\Delta$ for $k = 0,\dots,\mathtt{s}-1$, and let $V' = \Delta^{-1}V$ (here we can choose $\Delta = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$). The set $\mathcal{M}'$ satisfies the hypotheses of the case 3 hence $\lim\limits_{n\to\infty} d_{\text{columns}}(P_n(\omega)M_V) = \lim\limits_{n\to\infty} d_{\text{columns}}(\Delta M'_{\omega_1}\dots M'_{\omega_n}V') = 0$.

1.4 Proof of the converse assertion in Theorem 1.1

Now we suppose the existence of the uniform limit $V(\cdot) := \lim_{n\to\infty} \mathtt{N}(P_n(\cdot)V)$ and we want to check the conditions contained in one of the five cases involved in Theorem 1.1. Let $\mathcal{M}^2$ be the set of matrices MM' for $M, M' \in \mathcal{M}$, and let $\mathcal{U}$ (resp. $\mathcal{L}$) be the set of upper-triangular (resp. lower-triangular) matrices $M \in \mathcal{M} \cup \mathcal{M}^2$.

We first prove that $\mathcal{U}$ cannot contain a couple of matrices $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $A' = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$ such that $a \geq d$ and $a' < d'$: suppose that $\mathcal{U}$ contain such matrices let, for simplicity, $M_0 = A$ and $M_1 = A'$. One has $V(\overline{0}) = \lim_{n\to\infty} \mathtt{N}(A^n V)$, and this limit is also the normalized nonnegative right-eigenvector of A associated to its maximal eigenvalue, hence $V(\overline{0}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Similarly, $V(\overline{1})$ is colinear to $\begin{pmatrix} b' \\ d' - a' \end{pmatrix}$ (eigenvector of A') hence distinct from $V(\overline{0})$. Moreover, for fixed $N \in \mathbb{N}$

$$\begin{aligned} V(1^N\overline{0}) = \lim_{n\to\infty} \mathtt{N}(A'^N A^n V) &= \lim_{n\to\infty} \mathtt{N}(A'^N \mathtt{N}(A^n V)) \\ &= \mathtt{N}\left(A'^N V(\overline{0})\right) \\ &= V(\overline{0}). \end{aligned} \tag{8}$$

Since $1^N\overline{0}$ tends to $\overline{1}$ when $N \to \infty$, the inequality $V(\overline{0}) \neq V(\overline{1})$ contradicts the continuity of the map V. This proves that the couple of matrices $A, A' \in \mathcal{U}$ such that $a \geq d$ and $a' < d'$ do not exist. Similarly, the couple of matrices $A, A' \in \mathcal{L}$ such that $a \leq d$ and $a' > d'$ do not exist.

• Suppose that all the matrices in $\mathcal{U}$ satisfy $a \geq d$ and all the ones in $\mathcal{L}$ satisfy $a \leq d$. If $\mathcal{M}$ contains a matrix of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, it is necessarily an homothetic matrix. If it contains a matrix of the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, the square of this matrix is homothetic. So in both cases $\mathcal{M} \cup \mathcal{M}^2$ contains an homothetic matrix, let H. We use the same method as above: since the map V is continuous, the vector $\lim_{n\to\infty} \mathtt{N}(H^n V)$ must be equal to $\lim_{N\to\infty} \left(\lim_{n\to\infty} \mathtt{N}(H^N M H^n V)\right)$ for any $M \in \mathcal{M}$. But the first is $\mathtt{N}(V)$ and the second $\mathtt{N}(MV)$, hence V is an eigenvector of all the matrices in $\mathcal{M}$. Suppose now that $\mathcal{M}$ do not contain matrices of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ nor $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$: all the conditions of the case 1 are satisfied.

• Suppose that all the matrices in $\mathcal{U}$ satisfy $a < d$ and all the ones in $\mathcal{L}$ satisfy $a > d$; then the conditions of the case 2 are satisfied.

• Suppose that all the matrices $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{U}$ satisfy $a \geq d$ and all the matrices $A' = \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix} \in \mathcal{L}$ satisfy $a' > d'$. If there exists $A \in \mathcal{U}$, $A' \in \mathcal{L}$ and $M = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{M}$, the map V is discontinuous because

$$\lim_{n\to\infty} \mathtt{N}\left(A'^N M A^n \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \lim_{n\to\infty} \mathtt{N}\left(\begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix}^N \begin{pmatrix} \beta & 0 \\ \delta & \gamma \end{pmatrix} \begin{pmatrix} d & 0 \\ b & a \end{pmatrix}^n \begin{pmatrix} v_2 \\ v_1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

differs from $\lim_{n\to\infty} \mathtt{N}\left(A'^n \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)$ which is colinear to $\begin{pmatrix} a' - d' \\ c' \end{pmatrix}$. Hence, either $\mathcal{M}$ do not contain a matrix of the form $\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$ and we are in the case 3, or $\mathcal{U} = \emptyset$ and we are in the case 2, or $\mathcal{L} = \emptyset$ and we are in the case 1.

• The case when all the matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{U}$ satisfy $a < d$ and all the matrices $\begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix} \in \mathcal{L}$ satisfy $a' \leq d'$ is symmetrical to the previous, by using the set of matrices $\mathcal{M}' := \{\Delta^{-1} M \Delta \ ; \ M \in \mathcal{M}\}$ and the vector $V' = \Delta^{-1} V$, where $\Delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

2 Some properties of the Bernoulli convolutions in base $\beta > 1$

Given a real $\beta > 1$, an integer $\mathtt{d} > \beta$ and a $\mathtt{d}$-dimensional probability vector $\mathtt{p} := (\mathtt{p}_i)_{i=0}^{\mathtt{d}-1}$, the $\mathtt{p}$-*distributed* $(\beta, \mathtt{d})$-*Bernoulli convolution* is by definition the probability distribution $\mu_{\mathtt{p}}$ of the random variable X defined by

$$\forall \omega \in \mathcal{D}^{\mathbb{N}} := \ \{\mathbf{0}, \dots, \mathtt{d} - 1\}^{\mathbb{N}}, \ X(\omega) = \sum\nolimits_{k=1}^{\infty} \frac{\omega_k}{\beta^k},$$

where $\omega \mapsto \omega_k$ $(k = 1, 2, \cdots)$ is a sequence of i.i.d. random variables assuming the values $i = \mathbf{0}, \mathbf{1}, \dots, \mathtt{d} - 1$ with probability $\mathtt{p}_i$.

Denoting by $\overline{\omega}$ the sequence such that $\overline{\omega}_k = \mathtt{d} - 1 - \omega_k$ for any k, one has the relation $X(\omega) + X(\overline{\omega}) = \alpha := \dfrac{\mathtt{d} - 1}{\beta - 1}$. Hence, setting $\overline{\mathtt{p}}_i = \mathtt{p}_{\mathtt{d}-1-i}$ for any $i = \mathbf{0}, \mathbf{1}, \dots, \mathtt{d} - 1$, the following symmetry relation holds for any Borel set $B \subset \mathbb{R}$:

$$\mu_{\mathtt{p}}(B) = \mu_{\overline{\mathtt{p}}}(\alpha - B) \tag{9}$$

(notice that the support of $\mu_{\mathtt{p}}$ is a subset of $[0, \alpha]$).

The measure $\mu_{\mathtt{p}}$ also satisfy the following selfsimilarity relation: denoting by σ the shift on $\mathcal{D}^{\mathbb{N}}$ one has – for any Borel set $B \subset \mathbb{R}$

$$X(\omega) \in B \Leftrightarrow X(\sigma\omega) \in (\beta B - \omega_0)$$

hence, using the independance of the random variables $\omega \mapsto \omega_k$,

$$\mu_{\mathtt{p}}(B) = \sum_{k=0}^{\mathtt{d}-1} \mathtt{p}_k \cdot \mu_{\mathtt{p}}(\beta B - k) \quad \text{for any Borel set } B \subset \mathbb{R} \tag{10}$$

and in particular

$$\mu_{\mathtt{p}}(B) = \mathtt{p}_0 \cdot \mu_{\mathtt{p}}(\beta B) \quad \text{if } \beta B \subset [0,1]. \tag{11}$$

The following proposition is proved in [3, Theorem 2.1 and Proposition 5.4] in case the probability vector $\mathtt{p}$ is uniform:

Proposition 2.1 *The 1-periodic map $H :]-\infty, 0] \to \mathbb{R}$ defined by*

$$H(x) = (\mathtt{p}_0)^x \cdot \mu_{\mathtt{p}}\left([0, \beta^x[\right)$$

is continuous and a.e. differentiable. Moreover H is not differentiable on a certain continuum of points if β is an irrational Pisot number or an integer and – in this latter case – if β do not divide $\mathtt{d}$.

Let us give also the matricial form of the relation (10) (from [7, §2.1]). We define the (finite or countable) set $\mathcal{I}_{(\beta,\mathtt{d})} = \{0 = \mathtt{i}_0, \mathtt{i}_1, \cdots\}$ as follows (where $\mathcal{B}$ is the alphabet $\{0, 1, \ldots, \mathtt{b}-1\}$ such that $\mathtt{b} - 1 < \beta \leq \mathtt{b}$):

Definition 2.2 *$\mathcal{I}_{(\beta,\mathtt{d})}$ is the set of $i \in]-1, \alpha[$ for which there exists $-1 < i_1, \cdots, i_n < \alpha$ with $0 \triangleright i_1 \triangleright \cdots \triangleright i_n \triangleright i$, where $x \triangleright y$ means that exists $(\varepsilon, k) \in \mathcal{B} \times \mathcal{D}$ such that $y = \beta x + (\varepsilon - k)$.*

Let $\varepsilon \in \mathcal{B}$; the entries of the matrix M_ε are – for the row index i and the column index j, with $\mathtt{i}_i, \mathtt{i}_j \in \mathcal{I}_{(\beta,\mathtt{d})}$,

$$M_\varepsilon(i,j) = \begin{cases} \mathtt{p}_k & \text{if } k = \varepsilon + \beta \mathtt{i}_i - \mathtt{i}_j \in \mathcal{D} \\ 0 & \text{otherwise.} \end{cases}$$

Setting $\mathbb{R}_\varepsilon(x) = \dfrac{x+\varepsilon}{\beta}$ for any $\varepsilon \in \mathcal{B}$ and $x \in \mathbb{R}$, we have the following

PROPOSITION 2.3 *([7, Lemma 2.2]) If $\mathcal{I}_{(\beta,\mathtt{d})} = \{\mathtt{i}_0, \cdots, \mathtt{i}_{\mathtt{r}-1}\}$ then, for any Borel set $B \subset [0,1]$ and any $\varepsilon \in \mathcal{B}$ such that ${\mathbb{R}_\varepsilon}^{-1}(B) \subset [0,1]$,*

$$\begin{pmatrix} \mu_{\mathtt{p}}(B+\mathtt{i}_0) \\ \vdots \\ \mu_{\mathtt{p}}(B+\mathtt{i}_{\mathtt{r}-1}) \end{pmatrix} = M_\varepsilon \begin{pmatrix} \mu_{\mathtt{p}}\Big({\mathbb{R}_\varepsilon}^{-1}(B)+\mathtt{i}_0\Big) \\ \vdots \\ \mu_{\mathtt{p}}\Big({\mathbb{R}_\varepsilon}^{-1}(B)+\mathtt{i}_{\mathtt{r}-1}\Big) \end{pmatrix}.$$

REMARK 2.4 *The finiteness of $\mathcal{I}_{(\beta,\mathtt{d})}$ is assured, according to [7, §2.2], if β is an irrational Pisot number or an integer.*

We shall use also the probability distribution of the fractionnal part of the random variable X, that we denote by $\mu^*_{\mathtt{p}}$. Suppose that α belongs to $]1,2[$, or equivalently that $\beta < \mathtt{d} < 2\beta - 1$. Then $\mu^*_{\mathtt{p}}$ – which has support $[0,1]$ – satisfy the following relation for any Borel set $B \subset [0,1]$:

$$\mu^*_{\mathtt{p}}(B) = \mu_{\mathtt{p}}(B) + \mu_{\mathtt{p}}(B+1)$$

and, if $B \subset [\alpha-1,1]$,

$$\mu^*_{\mathtt{p}}(B) = \mu_{\mathtt{p}}(B). \tag{12}$$

The following proposition points out that in certain cases, the restriction of $\mu_{\mathtt{p}}$ (or $\mu^*_{\mathtt{p}}$) to the interval $[\alpha-1,1]$ is "representative" of $\mu_{\mathtt{p}}$ itself.

PROPOSITION 2.5 *Suppose $\beta < \mathtt{d} \leq \beta + 1 - \dfrac{1}{\beta}$.*

(i) The interval $]0,\alpha[$ is the reunion of $I_k := \left[\dfrac{1}{\beta^{k+1}}\ ,\ \dfrac{1}{\beta^k}\right]$ and $I'_k := \left[\alpha - \dfrac{1}{\beta^k}\ ,\ \alpha - \dfrac{1}{\beta^{k+1}}\right]$ for $k \in \mathbb{N} \cup \{0\}$

(ii) Let $B \subset \mathbb{R}$ be a Borel set . If $B \subset I_k$ (or equivalently if $\alpha - B \subset I'_k$), then $\beta^k B$ and $\alpha - \beta^k B$ are two subsets of $[\alpha-1,1]$ such that

$$\begin{aligned} \mu_{\mathtt{p}}(B) &= \mathtt{p}_0^k \cdot \mu^*_{\mathtt{p}}(\beta^k B) \\ \mu_{\mathtt{p}}(\alpha - B) &= \mathtt{p}_{\mathtt{d}-1}^k \cdot \mu^*_{\mathtt{p}}(\alpha - \beta^k B). \end{aligned}$$

Proof. (i) The hypothesis on $\mathtt{d}$ implies $\alpha < 2$ hence $]0,\alpha[$ is the reunion of $]0,1]$ and $[\alpha-1,\alpha[$.

(ii) $B \subset I_k \Rightarrow \beta^k B \subset \left[\dfrac{1}{\beta}\ ,\ 1\right] \subset [\alpha-1,1]$. The equality $\mu_{\mathtt{p}}(B) = \mathtt{p}_0^k \cdot \mu^*_{\mathtt{p}}(\beta^k B)$ results from (11) and (12).

Since $\beta^k B \subset [\alpha-1,1]$ one has $\alpha - \beta^k B \subset [\alpha-1,1]$. The equality $\mu_{\mathtt{p}}(\alpha - B) = \mathtt{p}_{\mathtt{d}-1}^k \cdot \mu_{\mathtt{p}}(\alpha - \beta^k B)$ follows from (9), (11) and (12). ∎

3 Bernoulli convolution in Pisot quadratic bases

In this section $\beta > 1$ is solution of the equation $x^2 = ax + b$ (with integral a and b), and we suppose that the other solution belongs to]-1,0[. This implies $1 \le b \le a \le \beta - \dfrac{1}{\beta} < \beta < a+1$. Let $\mathtt{p} = (\mathtt{p}_0, \dots, \mathtt{p}_a)$ be a positive probability vector; the Bernoulli convolution $\mu_{\mathtt{p}}$ has support $[0,\alpha]$, where $\alpha = \dfrac{a}{\beta - 1}$ belongs to $]1,2[$. The condition in Proposition 2.5 is satisfied hence it is sufficient to study the Gibbs properties of $\mu_{\mathtt{p}}^*$ on its support $[0,1]$, to get the local properties of $\mu_{\mathtt{p}}$ on $[0,\alpha]$ (see [6] for the multifractal analysis of the weak Gibbs measures).

With the notations of the previous subsection one has $\mathcal{B} = \mathcal{D} = \{0,\dots,a\}$,
$\mathcal{I}_{(\beta,a+1)} = \{0, 1, \beta - a\}$ and – for any $\varepsilon \in \mathcal{B}$

$$M_\varepsilon = \begin{pmatrix} \mathtt{p}_\varepsilon & \mathtt{p}_{\varepsilon-1} & 0 \\ 0 & 0 & \mathtt{p}_{a+\varepsilon} \\ \mathtt{p}_{b+\varepsilon} & \mathtt{p}_{b+\varepsilon-1} & 0 \end{pmatrix},$$

where, by convention, $\mathtt{p}_i = 0$ if $i \notin \mathcal{D}$.

Notice that the intervals $\mathbb{R}_\varepsilon([0,1])$ do not make a partition of $[0,1]$ for $\varepsilon \in \mathcal{B}$ but, setting

$$\mathbb{S}_\varepsilon := \begin{cases} \mathbb{R}_\varepsilon & \text{for } 0 \le \varepsilon \le a-1 \\ \mathbb{R}_a \circ \mathbb{R}_{\varepsilon - a} & \text{for } a \le \varepsilon \le a+b-1 \end{cases}$$

the intervals $\mathbb{S}_\varepsilon([0,1])$ make such a partition for $\varepsilon \in \mathcal{A} := \{0, \cdots, a+b-1\}$.

Theorem 3.1 *The measure $\mu_{\mathtt{p}}^*$ is weak Gibbs w.r.t. $\{\mathbb{S}_\varepsilon\}_{\varepsilon=0}^{a+b-1}$ if and only if $\mathtt{p}_0^2 \ge \mathtt{p}_a \, \mathtt{p}_{b-1}$ and $\mathtt{p}_0 \, \mathtt{p}_{a-b+1} \ge {\mathtt{p}_a}^2$.*

Proof. The n-step potential of $\mu_{\mathtt{p}}^*$ can be computed by means of the matrices

$$M_\varepsilon^* := \begin{cases} M_\varepsilon & \text{for } 0 \le \varepsilon \le a-1 \\ M_a M_{\varepsilon-a} & \text{for } a \le \varepsilon \le a+b-1. \end{cases}$$

Indeed by applying Proposition 2.3 to the sets $B = [\![\xi_1 \dots \xi_n]\!]$ and $B' = [\![\xi_2 \dots \xi_n]\!]$, one has

$$\exp(\phi_n(\xi)) = \log \left(\frac{(1 \;\; 1 \;\; 0)\, M_{\xi_1}^* \dots M_{\xi_n}^* V}{(1 \;\; 1 \;\; 0)\, M_{\xi_2}^* \dots M_{\xi_n}^* V} \right), \quad \text{where} \quad V := \begin{pmatrix} \mu_{\mathtt{p}}([0,1]) \\ \mu_{\mathtt{p}}([0,1]+1) \\ \mu_{\mathtt{p}}([0,1]+\beta-a) \end{pmatrix}. \tag{13}$$

Now the matrices M^*_ε are 3×3 and we shall use 2×2 ones. The matrices defined – for any $\varepsilon\in\mathcal{A}':=\{0,\dots,2a\}$ – by

$$M^\star_\varepsilon := \begin{cases} M_0M_\varepsilon & \text{if } \varepsilon\le a\\ M_{\varepsilon-a} & \text{if } \varepsilon> a\end{cases}$$

satisfy the commutation relation $YM^\star_\varepsilon = P_\varepsilon Y$, where

$$Y:=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\quad\text{and}\quad P_\varepsilon := \begin{cases}\begin{pmatrix}\mathtt{p}_0\mathtt{p}_\varepsilon & \mathtt{p}_0\mathtt{p}_{\varepsilon-1}\\ \mathtt{p}_a\mathtt{p}_{b+\varepsilon} & \mathtt{p}_a\mathtt{p}_{b+\varepsilon-1}\end{pmatrix} & \text{if } \varepsilon\le a\\ \begin{pmatrix}\mathtt{p}_{\varepsilon-a} & \mathtt{p}_{\varepsilon-a-1}\\ 0&0\end{pmatrix} & \text{if } \varepsilon> a.\end{cases}$$

Let $\xi\in\mathcal{A}^{\mathbb{N}}$ such that $\sigma\xi\ne\overline{0}$. There exists an integer $k\ge 0$ and $\varepsilon\in\mathcal{A}\setminus\{0\}$ such that

$$M_{\xi_2}\dots M_{\xi_{k+2}} = {M_0}^k M_\varepsilon.$$

One can associate to the sequence ξ, the sequence $\zeta\in\mathcal{A}'^{\mathbb{N}}$ such that

$$\forall n\ge k+4,\ \exists k(n)\in\mathbb{N},\quad M^*_{\xi_{k+3}}\dots M^*_{\xi_n} = M^\star_{\zeta_1}\dots M^\star_{\zeta_{k(n)}}\quad\text{or}\quad M^\star_{\zeta_1}\dots M^\star_{\zeta_{k(n)}}M_0.$$

Now – according to (13) and the commutation relation

$$n\ge k+4 \Rightarrow \exp(\phi_n(\xi)) = \frac{(\,1\quad 1\quad 0\,)\,M^*_{\xi_1}M_0^kQ_\varepsilon\mathtt{N}(P_{\zeta_1}\dots P_{\zeta_{k(n)}}W)}{(\,1\quad 1\quad 0\,)\,M_0^kQ_\varepsilon\mathtt{N}(P_{\zeta_1}\dots P_{\zeta_{k(n)}}W)} \tag{14}$$

where $Q_\varepsilon := \begin{pmatrix}\mathtt{p}_\varepsilon & \mathtt{p}_{\varepsilon-1}\\ 0&0\\ \mathtt{p}_{b+\varepsilon} & \mathtt{p}_{b+\varepsilon-1}\end{pmatrix}$ and $W=YV$ or YM_0V.

If $\mathtt{p}_0\ \mathtt{p}_{a-b+1}\ge {\mathtt{p}_a}^2$, the uniform convergence – on $\mathcal{A}'^{\mathbb{N}}$ – of the sequence $\mathtt{N}(P_{\zeta_1}\dots P_{\zeta_k}YV)$ and $\mathtt{N}(P_{\zeta_1}\dots P_{\zeta_k}YM_0V)$ to the same vector $V(\zeta)$ is insured by Theorem 1.1 and Corollary 1.2. When $n\to\infty$ the numerator in (14) converges to $V_1(\xi):=(\,1\quad 1\quad 0\,)\,M^*_{\xi_1}M_0^kQ_\varepsilon V(\zeta)$, and the denominator to $V_2(\xi):=(\,1\quad 1\quad 0\,)\,M_0^kQ_\varepsilon V(\zeta)$; this convergence is uniform on each cylinder $[\![\varepsilon' 0^k\varepsilon]\!]$. Since the first entrie in $Q_\varepsilon V(\zeta)$ is at least $\min\{\mathtt{p}_\varepsilon,\mathtt{p}_{\varepsilon-1}\}>0$, $V_1(\xi)$ and $V_2(\xi)$ are positive and consequently $\phi_n(\xi)$ converges uniformly to $\log\dfrac{V_1(\xi)}{V_2(\xi)}$. This is also true on any finite reunion of such cylinders; let us denote by $X(k_0)$ the reunion of $[\![\varepsilon' 0^k\varepsilon]\!]$ for $k<k_0, \varepsilon\in\mathcal{A}\setminus\{0\}$ and $\varepsilon'\in\mathcal{A}$; then

$$\forall\eta>0,\ \exists n_0\in\mathbb{N},\ n\ge n_0 \text{ and } \xi\in X(k_0)\Rightarrow\left|\phi_n(\xi)-\log\frac{V_1(\xi)}{V_2(\xi)}\right|\le\eta. \tag{15}$$

We consider now a sequence $\xi \in \mathcal{A}^{\mathbb{N}} \setminus X(k_0)$. By using the left and right eigenvectors of M_0 – for the eigenvalue $\mathtt{p}_0$ – we obtain

$$\lim_{k\to\infty} A_k = \lambda_0 \begin{pmatrix} \mathtt{p}_0^2 - \mathtt{p}_a\mathtt{p}_{b-1} & 0 & 0 \\ \mathtt{p}_a\mathtt{p}_b & 0 & 0 \\ \mathtt{p}_0\mathtt{p}_b & 0 & 0 \end{pmatrix} \quad \text{where } \lambda_0 > 0,\ A_k := \begin{cases} \mathtt{p}_0^{-k}{M_0}^k & \text{if } \mathtt{p}_a\ \mathtt{p}_{b-1} < {\mathtt{p}_0}^2 \\ k^{-1}\mathtt{p}_0^{-k}{M_0}^k & \text{if } \mathtt{p}_a\ \mathtt{p}_{b-1} = {\mathtt{p}_0}^2. \end{cases}$$

The entries $\mathtt{p}_a\mathtt{p}_b$ and $\mathtt{p}_0\mathtt{p}_b$ being positive, there exists $\lambda(\varepsilon') > 0$ such that

$$\lim_{k\to\infty} (\,1 \quad 1 \quad 0\,)\, M^*_{\varepsilon'} A_k Q_\varepsilon = \lambda(\varepsilon')\,(\,\mathtt{p}_\varepsilon \quad \mathtt{p}_{\varepsilon-1}\,)\,.$$

Moreover the convergence of $(\,1 \quad 1 \quad 0\,)\, M^*_{\varepsilon'} A_k Q_\varepsilon \begin{pmatrix} x \\ y \end{pmatrix}$ to $\lambda(\varepsilon')(\mathtt{p}_\varepsilon x + \mathtt{p}_{\varepsilon-1} y)$ is uniform on the set of normalized nonnegative column-vectors $\begin{pmatrix} x \\ y \end{pmatrix}$. Similarily, there exists $\lambda_1 > 0$ such that $(\,1 \quad 1 \quad 0\,)\, A_k Q_\varepsilon \begin{pmatrix} x \\ y \end{pmatrix}$ converges uniformly to $\lambda_1(\mathtt{p}_\varepsilon x + \mathtt{p}_{\varepsilon-1} y)$. Both limits are positive if $\varepsilon \neq 0$, implying that the ratio converges uniformly to $\dfrac{\lambda(\varepsilon')}{\lambda_1}$. Hence, using (14) one can choose k_0 such that – if we assume $\xi \in \mathcal{A}^{\mathbb{N}} \setminus X(k_0)$ and $\sigma\xi \neq \overline{0}$

$$n \geq k_0 + 4 \Rightarrow \left| \phi_n(\xi) - \log \frac{\lambda(\varepsilon')}{\lambda_1} \right| \leq \eta. \tag{16}$$

The uniform convergence of $\phi_n(\xi)$ on $\mathcal{A}^{\mathbb{N}}$ follows from (15) and (16) since, in the remaining case $\sigma\xi = \overline{0}$ one has $\lim_{n\to\infty} \phi_n(\xi)) = \log \dfrac{\lambda(\xi_1)}{\lambda_1}$.

Conversely, suppose $\mathtt{p}_a\ \mathtt{p}_{b-1} > \ {\mathtt{p}_0}^2$. If $\mu^*_{\mathtt{p}}$ is weak Gibbs w.r.t. $\{\mathbb{S}_\varepsilon\}_{\varepsilon=0}^{\mathtt{s}-1}$ then, from (1) and (2) one has $\phi_n(\xi) = o(n)$ for any $\xi \in \mathcal{A}^{\mathbb{N}}$. But this is not true: $\phi_{2n+1}(1\overline{0}) \sim n \log \dfrac{\mathtt{p}_a\mathtt{p}_{b-1}}{{\mathtt{p}_0}^2}$.

Suppose now $\mathtt{p}_0\ \mathtt{p}_{a-b+1} < {\mathtt{p}_a}^2$. If $b = 1$ we have $\mathtt{p}_0 < \mathtt{p}_a$ hence $\mathtt{p}_a\ \mathtt{p}_{b-1} > \ {\mathtt{p}_0}^2$; that is, we are in the previous case. If $b \neq 1$, $\mu^*_{\mathtt{p}}$ is no more weak Gibbs w.r.t. $\{\mathbb{S}_\varepsilon\}_{\varepsilon=0}^{\mathtt{s}-1}$ because there exists a contradiction between the limit in (1) and the following:

$$\lim_{n\to\infty} \left(\frac{\mu^*_{\mathtt{p}}[\![(0(a-b+1))^n 1^n]\!]}{\mu^*_{\mathtt{p}}[\![(0(a-b+1))^n]\!] \cdot \mu^*_{\mathtt{p}}[\![1^n]\!]} \right)^{1/n} = \frac{\mathtt{p}_0\mathtt{p}_{a-b+1}}{{\mathtt{p}_a}^2} < 1.$$

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